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Affine geometry designs, polarities, and Hamada's conjecture

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To Marialuisa J. de Resmini on the occasion of her retirement

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ABSTRACT

In a recent paper, two of the authors used polarities in $PG(2d-1,p)$ ($p \geqslant 2$ prime, $d \geqslant 2$) to construct non-geometric designs having the same parameters and the same p-rank as the geometric design $PG_d(2d,p)$ having as blocks the d-subspaces in the projective space $PG(2d,p)$, hence providing the first known infinite family of examples where projective geometry designs are not characterized by their p-rank, as it is the case in all known proven cases of Hamada's conjecture. In this paper, the construction based on polarities is extended to produce designs having the same parameters, intersection numbers, and 2-rank as the geometric design $AG_{d+1}(2d+1,2)$ of the $(d+1)$-subspaces in the binary affine geometry $AG(2d+1,2)$. These designs generalize one of the four non-geometric self-orthogonal 3-$(32,8,7)$ designs of 2-rank 16 (V.D. Tonchev, 1986 [12]), and provide the only known infinite family of examples where affine geometry designs are not characterized by their rank.

1. Introduction

Let X be a set of v *points*, and B be a collection of k-subsets of X called *blocks*. Then $D = (X, B)$ is a t-(v,k,λ) *design* or *block design* if every t-subset of X is contained in exactly λ blocks. Two designs $D_1 = (X_1, B_1)$ and $D_2 = (X_2, B_2)$ are *isomorphic* if there is a bijection from X_1 to X_2 which maps B_1 to B_2. The *automorphism group* of D is the subgroup of $\mathrm{Sym}(X)$ whose action on X preserves B.

If v is divisible by k, a *parallel class* of D is a set of v/k blocks which partition X. If B can be partitioned into disjoint parallel classes, then D is said to be *resolvable*, and any particular partition is called a *resolution*.

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The *incidence matrix* of D is a $v \times b$ matrix $A = (a_{ij})$ where $a_{ij} = 1$ if point i of X is contained in block j of B, and 0 otherwise. The rows of A^T are the incidence vectors of the blocks of D. These may be taken to span a linear error-correcting code called the *block code* of D.

Classical examples of designs are obtained from finite geometries. We construct these geometries using the n-dimensional vector space V over a finite field $GF(q)$. The $(n-1)$-dimensional *projective geometry* $PG(n-1,q)$ over $GF(q)$ has as points the 1-dimensional subspaces of V. Its lines are the 2-dimensional subspaces of V, and in general the d-dimensional projective subspaces are the $(d+1)$-dimensional subspaces of V. Taking the d-dimensional projective subspaces of $PG(n-1,q)$ as blocks, we obtain a design denoted by $PG_d(n-1,q)$ with parameters

$$v = \frac{q^n - 1}{q-1}, \qquad k = \frac{q^{d+1}-1}{q-1}, \qquad \lambda = \begin{bmatrix} n-1 \\ d-1 \end{bmatrix}_q,$$

where $\left[\begin{smallmatrix} n-1 \\ d-1 \end{smallmatrix}\right]_q$ is the Gaussian coefficient given by

$$\begin{bmatrix} n-1 \\ d-1 \end{bmatrix}_q = \frac{(q^{n-1}-1)(q^{n-2}-1)\cdots(q^{n-d+1}-1)}{(q^{d-1}-1)(q^{d-2}-1)\cdots(q-1)}.$$

Similarly, the n-dimensional *affine geometry* $AG(n,q)$ over $GF(q)$ has as points the vectors of V. Its lines are the 1-dimensional subspaces of V and their cosets, and in general the d-dimensional affine subspaces are the d-dimensional subspaces of V and their cosets. Taking the d-dimensional affine subspaces of $AG(n,q)$ as blocks, one obtains a design denoted by $AG_d(n,q)$ with parameters

$$v = q^n, \qquad k = q^d, \qquad \lambda = \begin{bmatrix} n-1 \\ d-1 \end{bmatrix}_q.$$

This design is resolvable: the set of all cosets of a vector subspace forms a natural parallel class.

For further terminology and results on designs, see [2]. For specific information on geometric designs, see [1], especially Chapter 3.

Let q be a prime power and $\Pi = PG_d(2d,q)$, $d \geqslant 2$. Let $H \simeq PG(2d-1,q)$ be a hyperplane in $PG(2d,q)$, and let α be a polarity [7] of H. A block B of Π is either contained in H or intersects H in a $(d-1)$-subspace. It was proved by Jungnickel and Tonchev in [8] that replacing each $(d-1)$-subspace $B \cap H$ by $\alpha(B \cap H)$ yields a design $\alpha(\Pi)$ having the same parameters and block intersection numbers as $PG_d(2d,q)$. In addition, if q is prime, $\alpha(\Pi)$ has the same q-rank as $PG_d(2d,q)$, thus providing a counterexample to the "only if" part of Hamada's conjecture [6], which states that a design with the parameters of $PG_d(n,q)$ or $AG_d(n,q)$ is geometric if and only if it has minimum q-rank among all designs with the given parameters.

It was proved recently by Munemasa and the third author [9] that the block graph of the design obtained from $PG_d(2d,q)$ via the construction of Jungnickel and Tonchev [8], where two blocks are adjacent if they share $(q^d-1)/(q-1)$ points, is a distance-regular graph isomorphic to the twisted Grassmann graph discovered by van Dam and Koolen [4].

In this paper, we show that the construction from [8] can be extended to yield an infinite family of non-geometric designs with the same parameters, intersection numbers, and 2-rank as the affine geometry design $\mathcal{A} = AG_{d+1}(2d+1,2)$ having as blocks the $(d+1)$-dimensional subspaces of the binary affine space $AG(2d+1,2)$, for any $d \geqslant 2$. This provides the first known infinite family of counterexamples to the "only if" part of Hamada's conjecture in the affine case. This work was motivated by the smallest example ($d=2$), which corresponds to one of the four non-geometric self-orthogonal 3-(32, 8, 7) designs [12] of 2-rank 16.

Hamada's conjecture is of interest for two key reasons. The block codes of geometric designs admit majority logic decoding [10,11]. The effectiveness of this method depends on the dimension of the code, which is equal to the p-rank of the corresponding design. Thus, Hamada's conjecture identifies the geometric designs, due to their minimum p-rank, as the best choices for use in constructing error-correcting codes which admit majority-logic decoding. The minimum p-rank also provides a simple invariant for identification of geometric designs (among all other designs with the same parameters) while avoiding the problem of design isomorphism, which is notoriously difficult [3, Remark VII.6.6].

2. Polarities in binary affine space

Let $\mathcal{A} = AG_{d+1}(2d+1,2)$. Then $\mathcal{A}$ is a 3-(v,k,λ_3) design with parameters

$$v = 2^{2d+1}, \qquad k = 2^{d+1}, \qquad \lambda_3 = \frac{(2^{2d-1}-1)\cdots(2^{d+1}-1)}{(2^{d-1}-1)\cdots(2-1)} = \begin{bmatrix} 2d-1 \\ d-1 \end{bmatrix}_2. \tag{1}$$

The number of blocks containing a pair of points of $\mathcal{A}$ is given by

$$\lambda_2 = \frac{2^{2d+1}-2}{2^{d+1}-2}\lambda_3 = \begin{bmatrix} 2d \\ d \end{bmatrix}_2,$$

while the number of blocks containing a single point of $\mathcal{A}$ is equal to

$$\lambda_1 = \frac{2^{2d+1}-1}{2^{d+1}-1}\lambda_2 = \begin{bmatrix} 2d+1 \\ d+1 \end{bmatrix}_2.$$

Let X denote the point set of $\mathcal{A}$, and let $\bar{0} \in X$ be the point of $AG(2d+1,2)$ that corresponds to the zero vector in $GF(2)^{2d+1}$. The collection of blocks of $\mathcal{A}$ which contain $\bar{0}$ induces on $X \setminus \{\bar{0}\}$ a 2-$(2^{2d+1}-1, 2^{d+1}-1, \left[\begin{smallmatrix} 2d-1 \\ d-1 \end{smallmatrix}\right]_2)$ design D_0 isomorphic to $PG_d(2d,2)$.

Let $H \subset X$ be a set of 2^{2d} points such that $\bar{0} \in H$, and H is a $2d$-subspace of $AG(2d+1,2)$. Then H is a hyperplane of $\mathcal{A}$. Note that H is a linear subspace of $\mathcal{A}$. A block B which intersects H in a d-dimensional affine subspace will be called a *cross* block. Note that $|B \cap H| = |B \setminus H| = 2^d$. We will write $B = B_{out} \cup B_{in}$, where $B_{out} = B \setminus H$ and $B_{in} = B \cap H$. We refer to B_{out} as the *outer* part of B, and B_{in} as the *inner* part. Note that $B_{out} \cap B_{in} = \emptyset$.

All blocks of $\mathcal{A}$ have 2^d translates (or cosets) in the group of translations of $\mathcal{A}$. For a cross block B, these translates may be written as $\{B + h_i \mid h_i \in H\}$. That is, the group of translations of H is enough to produce all translates of B within $\mathcal{A}$. Note that for any cross block B, any translate also intersects H in exactly 2^d points.

In addition, for any cross block $B = B_{out} \cup B_{in}$ of $\mathcal{A}$, B_{out} is a translate of B_{in} by an element of $X \setminus H$. As a result, the set $\{B' \setminus H \mid B' \cap H = B_{in}\}$ consists of a partition of $X \setminus H$ into translates. Similarly, $\{B' \cap H \mid B' \setminus H = B_{out}\}$ partitions H into translates.

With this in mind, we present the following construction, which extends the construction of [8] to certain binary affine geometries.

Construction 2.1. With H as above, let α be a permutation of the affine d-subspaces through $\bar{0}$, of the affine space $AG(2d,2)$ induced on H.

Using α, we make the following alterations to the blocks of $\mathcal{A}$:

- If B is a block such that $B \subset H$ or $B \cap H = \emptyset$, we leave B unchanged.
- If $|B \cap H| = 2^d$ and $\bar{0} \in B$, we replace the inner part B_{in} of B by $\alpha(B_{in}) = \alpha(B \cap H)$.
- If $|B \cap H| = 2^d$ and $\bar{0} \notin B$, there is a block B_1 such that $\bar{0} \in B_1$, $|B_1 \cap H| = 2^d$, and $B \cap H$ is a translate (or coset) of $B_1 \cap H$ in the group of translations of H, by considering H as a $2d$-dimensional vector space. Let $\{h_1 = \bar{0}, h_2, \ldots, h_{2^d}\}$ be 2^d distinct elements of H such that:
  – each coset of B_1 is represented exactly once in the set $\{B_1 + h_i \mid i = 1, \ldots, 2^d\}$, and
  – each coset of $\alpha(B_1 \cap H)$ is represented exactly once in the set $\{\alpha(B_1 \cap H) + h_i \mid i = 1, \ldots, 2^d\}$.
  Such a set of h_i exists by Hall's matching theorem [5], see Lemma 2.2 below.
  Let $B_2, B_3, \ldots, B_{2^d}$ be all other blocks such that $B_i \cap H = B_1 \cap H$. Note that the outer part of B_i is a translate of the outer part of B_1 by an element $h \in H$, and that $\bar{0} \in B_i$ for each $1 \leqslant i \leqslant 2^d$. In particular, each coset of B_i may be represented as $B_i + h_j$ for some $1 \leqslant j \leqslant 2^d$. We replace the part of B_i equal to $B_i \cap H$ with $\alpha(B_i \cap H)$, for $1 \leqslant i \leqslant 2^d$. For the coset of B_i equal to $B_i + h_i$, we replace the part equal to $(B_i + h_i) \cap H$ with $\alpha(B_i \cap H) + h_i$.

Notice that this construction effectively permutes the inner parts of all cross blocks, including those which are translates. The construction guarantees that the multiset of inner portions of cross blocks is preserved.

For a cross block $B = B_{out} \cup B_{in}$, we will write $\alpha(B) = B_{out} \cup \alpha(B_{in})$ to represent the "distorted" block produced by the construction. Note that writing the block this way makes sense, because the construction does not touch the outer parts of cross blocks.

The following technical lemma is necessary to show the correctness of the construction. It will also provide the basis for a related construction over any finite field. Note that our original definition of a cross block extends naturally to q-ary affine geometries: any block which intersects a hyperplane H in a d-dimensional affine space is still a *cross block*.

Lemma 2.2. *Let $\mathcal{A} = AG_{d+1}(2d+1, q)$ and H be a hyperplane of $\mathcal{A}$ through $\bar{0}$, and let α be a permutation of the affine d-subspaces of H which contain $\bar{0}$.*

Let B_1 be a cross block of $\mathcal{A}$ through $\bar{0}$. Then there exists a set $\{h_1 = \bar{0}, h_2, \ldots, h_{q^d}\}$ of distinct elements of H such that:

- *each coset of B_1 is represented exactly once in the set*
$$\{B_1 + h_i \mid i = 1, \ldots, q^d\},$$
and
- *each coset of $\alpha(B_1 \cap H)$ is represented exactly once in the set*
$$\{\alpha(B_1 \cap H) + h_i \mid i = 1, \ldots, q^d\}.$$

Proof. First, as mentioned above, it is possible to find all translates of B_1, and all translates of $\alpha(B_1 \cap H)$ respectively using only elements of H. This holds for affine geometry designs over any finite field.

Let $G = (V_1 \cup V_2, E)$ be a bipartite multigraph with V_1 being the q^d translates of B_1 shifted by elements of H, and V_2 being the q^d translates of $\alpha(B_1 \cap H)$ by elements in H. We place an edge $\{x, y\}$ if there exists an $h \in H$ such that $x = B_1 + h$ and $y = \alpha(B_1 \cap H) + h$. Finding a set of h_i as described is equivalent to finding a perfect matching in G.

For each coset of B_1 or of $\alpha(B_1 \cap H)$, there are q^d values of h which produce the same coset. For any $X \subseteq V_1$, there are $q^d \cdot |X|$ vectors h which produce some coset in X. Similarly, for the cosets in $N(X)$, there are $q^d \cdot |N(X)|$ vectors which produce some coset in $N(X)$, where $N(X)$ represents the set of neighbors of X in V_2. As each vector corresponds to a distinct edge, we have $q^d|X| = q^d|N(X)|$, and so $|X| = |N(X)|$. Thus by Hall's matching theorem [5], a perfect matching exists in G.

Specializing with $q = 2$, we obtain the result necessary for Construction 2.1. $\square$

Theorem 2.3. *The collection of blocks $\alpha(\mathcal{A})$ obtained from $\mathcal{A}$ via Construction* 2.1 *is a resolvable* 3*-design with the same parameters as $\mathcal{A} = AG_{d+1}(2d+1, 2)$.*

Proof. All blocks in $\alpha(\mathcal{A})$ have size 2^{d+1}, because α only permutes d-subspaces within H.

The resulting structure is resolvable by construction. Consider a parallel class P of blocks in $\mathcal{A}$. If any block of P is contained entirely in H, then 2^{d-1} blocks of P are entirely contained in H, and the rest are disjoint from H. These blocks are untouched by the construction, and so remain a parallel class. On the other hand, if any block of P intersects H in 2^d points, then all blocks of P do so. In this case, recall that P consists of all cosets of the block $B \in P$ containing $\bar{0}$. The construction distorts B and its cosets in such a way that the distorted versions of the blocks of P remain pairwise disjoint, and thus form a parallel class. Thus $\alpha(\mathcal{A})$ is resolvable.

We must check that Construction 2.1 does not change distinct blocks into the same block. Suppose B, B' are blocks of $\mathcal{A}$ both containing $\bar{0}$. It is clear from the construction that if $B \neq B'$, then $\alpha(B) \neq \alpha(B')$. Now we must consider cosets. Suppose B, B' are cross blocks containing $\bar{0}$. Write $B = B_{out} \cup B_{in}$ and $B' = B'_{out} \cup B'_{in}$. Then $\alpha(B) = B_{out} \cup \alpha(B_{in})$ and $\alpha(B') = B'_{out} \cup \alpha(B'_{in})$. Suppose $\alpha(B) + h = \alpha(B') + h'$ for some $h, h' \in H$. Then $B_{out} \cup \alpha(B_{in}) = (B'_{out} \cup \alpha(B'_{in})) + (h + h')$, and in particular $\alpha(B_{in}) = \alpha(B'_{in}) + (h + h')$. But both $\alpha(B_{in})$ and $\alpha(B'_{in})$ are vector subspaces, so $h + h' \in \alpha(B'_{in})$, and thus $\alpha(B_{in}) = \alpha(B'_{in})$. Thus B and B' have the same inner parts, and so h and h' were chosen as specified in the construction. If $h = h'$, then $B_{out} = B'_{out}$ and so $B = B'$. If $h \neq h'$, then $\alpha(B_{in}) + h \neq \alpha(B'_{in}) + h'$ by construction, and so $\alpha(B_{in}) \neq \alpha(B'_{in}) + (h + h')$, contradicting our previous argument. Thus $B + h \neq B + h'$. In either case, we see that this construction produces distinct blocks from the blocks of $\mathcal{A}$. Note that if h, h' were not chosen as in the construction, it would be possible to transform two distinct blocks into the same block.

Finally, we show that $\alpha(\mathcal{A})$ is a 3-design with the same value of λ_3. Consider a triple $T = \{x, y, z\}$ of distinct points of $AG(2d + 1, 2)$. We consider several cases:

- If $T \subset H$, then any block $B = B_{out} \cup B_{in}$ containing T has $T \subset B_{in}$. Because α permutes the inner parts of cross blocks, the number of cross blocks containing T is unchanged.
- Similarly, if $T \subset X \setminus H$, then the number of cross blocks containing T is unchanged.
- Suppose $\{x, y\} \subseteq H$ and $z \in X \setminus H$. Consider any d-dimensional vector subspace S of H containing $\{x, y\}$ and $\bar{0}$. Then among all cross blocks meeting H in S, exactly one contains z (because the outer parts of these blocks are translates which partition $X \setminus H$). There is a one-to-one correspondence between cross blocks of $\mathcal{A}$ containing S, and cross blocks of $\alpha(\mathcal{A})$ containing S. In $\alpha(\mathcal{A})$, the outer parts of each such block still partition $X \setminus H$. Thus the number of cross blocks containing both $\bar{0}$ and T is fixed.
  To account for cosets, suppose R is a d-dimensional vector subspace of H containing $\bar{0}$. Then $\{x, y\}$ is contained in a coset $R + h$ for some $h \in H$ if and only if $\{x + h, y + h\}$ is contained in R, so the argument remains the same for cosets.
- Similarly, suppose that $x \in H$ but $\{y, z\} \subseteq X \setminus H$. Let $B = B_{out} \cup B_{in}$ be a cross block of $\mathcal{A}$ containing $\bar{0}$ such that $\{y, z\} \subset B_{out}$. Let C be the set of cross blocks of $\mathcal{A}$ whose outer parts are equal to B_{out}. Then the inner parts of the blocks in C are translates of B_{in} which partition H. Thus exactly one such inner part contains x. The construction replaces the inner part of each block of C with a distinct coset of $\alpha(B_{in})$, and these cosets partition H. Thus exactly one of these distorted blocks contains $\{x, y, z\}$.
  To account for cosets, note that a cross block's outer part contains $\{y, z\}$ if and only if there is a translate of the block, through $\bar{0}$, whose outer part contains $\{y + h, z + h\}$.

Thus the number of blocks containing T is unchanged, and so $\alpha(\mathcal{A})$ is a 3-design with index λ_3. $\square$

We defined α to be a permutation of affine d-spaces through $\bar{0}$. Because we are working with binary geometries, each point $\neq 0$ of $\mathcal{A}$ may be identified with a unique point of the projective geometry $PG(2d, 2)$ induced on X. Each projective $(d - 1)$-space in the copy of $PG(2d, 2)$ induced on H may be uniquely extended to an affine d-space through $\bar{0}$ by simply adding $\bar{0}$ to the space. Note that if α is a polarity of the projective space $PG(2d - 1, 2)$ induced on H, then it permutes projective $(d - 1)$-spaces. Thus we may view α as a permutation of the affine d-spaces through $\bar{0}$ of H. In this case, we can obtain more detailed information about the properties of $\alpha(\mathcal{A})$.

Theorem 2.4. *If α is a polarity of the projective space $PG(2d - 1, 2)$ induced on H, then the design $\alpha(\mathcal{A})$ has the same intersection numbers as $\mathcal{A}$.*

Proof. Any two blocks of $\mathcal{A}$ are either disjoint or share 2^i points for some integer $1 \leqslant i \leqslant d$.

Let $B = B_{out} \cup B_{in}$ and $B' = B'_{out} \cup B'_{in}$ be cross blocks of $\mathcal{A}$, both containing $\bar{0}$. Construction 2.1 as applied to any block through $\bar{0}$ is equivalent to the construction of [8], and thus the intersection numbers of these blocks are unchanged. In particular, $|\alpha(B) \cap \alpha(B')| = |B \cap B'|$, and if $B \cap B' \neq \emptyset$, then $|\alpha(B_{in}) \cap \alpha(B'_{in})| = |B_{in} \cap B'_{in}| = 2^i$ for some $0 \leqslant i \leqslant d$.

Now we consider cosets. For $h \in H$, $|\alpha(B_{in}) \cap (\alpha(B'_{in}) + h)|$ is either 0, or exactly $|\alpha(B_{in}) \cap \alpha(B'_{in})|$. The cosets of $\alpha(B_{in}) \cap \alpha(B'_{in})$ shifted by elements of $\alpha(B_{in})$ partition $\alpha(B_{in})$, whereas the cosets of $\alpha(B_{in}) \cap \alpha(B'_{in})$ by any other elements of H are disjoint from $\alpha(B_{in})$.

For the outer parts, note that $X \setminus H$ is (the only) coset of H in X. Thus all of our previous arguments for inner parts apply to the outer parts as well. In particular, B_{out} and B'_{out} may be written as $S + k$ and $S' + k$ for some d-dimensional vector subspaces S, S' of H, and $k \in X \setminus H$. Thus,

$$\left|B_{out} \cap \left(B'_{out} + h\right)\right| = \left|(S+k) \cap \left(S' + k + h\right)\right| = \left|S \cap \left(S' + h\right)\right|,$$

and by the previous argument, these intersections have the same sizes as the intersections of inner parts. Consequently, $|B_{out} \cap (B'_{out} + h)|$ is either 0 or $|B_{out} \cap B'_{out}|$, where $|B_{out} \cap B'_{out}| = 2^i$ for some $0 \leqslant i \leqslant d$.

Thus, $|B \cap (B' + h)|$ is either 0, $|B_{in} \cap B'_{in}|$, $|B_{out} \cap B'_{out}|$, or $|B \cap B'|$. In any case, B and $B' + h$ are either disjoint, or intersect in 2^i points for some $0 \leqslant i \leqslant d$. We can actually make a stronger statement: B_{out} is a coset of B_{in} for any cross block of $\mathcal{A}$, and so $|B_{in} \cap B'_{in}| = |B_{out} \cap B'_{out}|$. Thus $|B \cap (B' + h)|$ has only three possible values: 0, $|B \cap B'|$, or $|B \cap B'|/2$.

Assume that $|B_{out} \cap B'_{out}| = 1$ or $|\alpha(B_{in}) \cap \alpha(B'_{in})| = 1$. In the design $\mathcal{A}$, we have $|B_{out} \cap B'_{out}| = 1$ if and only if $|B_{in} \cap B'_{in}| = 1$, because intersection numbers in $\mathcal{A}$ are even. Then $B_{in} \cap B'_{in} = \{\bar{0}\}$, and so $(B_{in} \setminus \{\bar{0}\}) \cap (B'_{in} \setminus \{\bar{0}\}) = \emptyset$. Since α is incidence-preserving, we have $|\alpha(B_{in}) \cap \alpha(B'_{in})| = 1$ as well. In addition, note that if $|B_{out} \cap B'_{out}| = 1$, then $|B_{out} \cap (B'_{out} + h)| = 1$ for all $h \in H$, and similarly for $|B_{in} \cap (B'_{in} + h)|$. Thus $|B_{out} \cap (B'_{out} + h)| = 1$ if and only if $|\alpha(B_{in}) \cap (\alpha(B'_{in}) + h)| = 1$, and so $|B \cap (B' + h)| = 2$.

Therefore, the set of intersection numbers of cross blocks and their cosets is the same as the set of intersection numbers of $\mathcal{A}$.

Finally, we consider a non-cross block B. The intersection of B with other non-cross blocks is obviously unchanged. The intersection of B with a cross block B' occurs entirely in either H or $X \setminus H$, thus it is either 0 or 2^i, for some $0 \leqslant i \leqslant d$. Note however that by their dimensions, no block of size 2^{d+1} contained entirely in H or entirely in $X \setminus H$ can intersect a space of size 2^d in only 1 point.

Thus, the block intersection numbers of $\alpha(\mathcal{A})$ are a subset of the block intersection numbers of $\mathcal{A}$. Blocks contained entirely in H do have all intersection numbers including 0 and 2^i for each $1 \leqslant i \leqslant d$. Consequently, the set of intersection numbers of blocks in $\mathcal{A}$ and $\alpha(\mathcal{A})$ are identical. □

Theorem 2.5. *If α is a polarity of the projective space $PG(2d-1, 2)$ induced on $H \setminus \{\bar{0}\}$, then the design $\alpha(\mathcal{A})$ has the same 2-rank as $\mathcal{A}$, but is not isomorphic to $\mathcal{A}$.*

Proof. Note that the block code of $\mathcal{A}$ is the Reed–Muller code $R(d, 2d+1)$ which has dimension 2^{2d} and is self-dual [1]. Thus the 2-rank of $\mathcal{A}$ is 2^{2d}.

From the intersection numbers, the block code $\mathcal{C}$ of $\alpha(\mathcal{A})$ is self-orthogonal. Thus $\dim \mathcal{C} \leqslant 2^{2d}$, and so the 2-rank of $\alpha(\mathcal{A})$ is at most 2^{2d}. On the other hand, Construction 2.1 transforms the design D_0 of $\mathcal{A}$ into a design $\alpha(D_0)$ with the same parameters, but not isomorphic to $PG_d(2d, 2)$, and having 2-rank equal to 2^{2d} [8]. Hence, the 2-rank of $\alpha(\mathcal{A})$ is equal to 2^{2d}, and the design $\alpha(\mathcal{A})$ is not isomorphic to $\mathcal{A}$. □

The designs produced by Construction 2.1 provide an infinite family of examples of geometric designs, $AG_{d+1}(2d+1, 2)$, $d \geqslant 2$, which are not characterized as the unique designs with the given parameters and 2-rank. Thus, if the weak version of Hamada's conjecture (namely that the classical design $AG_{d+1}(2d+1, 2)$ has the minimum 2-rank among all designs with the same parameters) is true, it follows that for each $d \geqslant 2$ there is at least one other design, namely $\alpha(\mathcal{A})$, having the same parameters and the same (minimum) 2-rank. Hence we have obtained the first known infinite family of counteresamples to the strong version of the conjecture (the "only if" part) in the affine case.

Example 2.6. The smallest example of this construction corresponds to the design $\mathcal{A} = AG_3(5, 2)$ whose blocks are the 3-dimensional vector subspaces of a 5-dimensional binary vector space, and their cosets. The design $\mathcal{A}$ is a 3-(32, 8, 7) design with 620 blocks. We apply Construction 2.1 using the hyperplane $H = \langle 00001, 00010, 00100, 01000 \rangle$ and the orthogonal polarity α of $PG(4, 2)$. The 2-rank of both $\mathcal{A}$ and $\alpha(\mathcal{A})$ is 16.

The automorphism group of $\mathcal{A}$ is $A\Gamma L(5, 2)$ of order $2^{15} \cdot 3^2 \cdot 5 \cdot 7 \cdot 31$. It is 3-transitive on points and transitive on blocks. (See for example [2].) The automorphism group of $\alpha(\mathcal{A})$ has order $2^{15} \cdot 3^2 \cdot 5 \cdot 7$. It is point-transitive but not block-transitive.

To examine the block orbits of $\alpha(\mathcal{A})$, we view the points of $\mathcal{A}$ as elements of $F = GF(2^5)$. Thus 01000 represents w^2, where w is a primitive element of F. We identify each point with the exponent

i of its representation w^i, thus $3 = 00100, 4 = 00010, \dots, 31 = 10000$, and $0 = 00000$. In this notation, the automorphism group of $\alpha(\mathcal{A})$ is generated by the following eleven permutations found by computer with Magma:

$(0, 16, 28, 25, 13, 23, 24, 29, 30, 17, 19, 26)(1, 14, 9, 18, 3, 27, 21, 5, 2, 10, 20, 31)(4, 22, 6, 8, 7, 15)(11, 12)$
$(5, 25)(8, 10)(11, 16)(14, 31)(15, 18)(17, 23)(22, 29)(26, 27)$
$(1, 21, 2, 24)(3, 28, 7, 20)(4, 30, 6, 12)(5, 27, 14, 17)(8, 22)(9, 13)(10, 26, 15, 31)(11, 25, 29, 23)$
$(2, 12, 7)(3, 21, 30)(4, 6, 9)(5, 18, 14, 23, 29, 26)(8, 17, 31, 11, 25, 27)(10, 16)(15, 22)(19, 20, 28)$
$(5, 27)(8, 18)(10, 15)(11, 29)(14, 17)(16, 22)(23, 31)(25, 26)$
$(4, 9)(7, 24)(8, 14)(11, 26)(13, 28)(17, 18)(21, 30)(25, 29)$
$(4, 28)(5, 15, 22, 23)(7, 30)(8, 26, 14, 11)(9, 13)(10, 16, 31, 27)(17, 29, 18, 25)(21, 24)$
$(4, 21)(5, 25, 22, 29)(7, 13)(8, 31, 14, 10)(9, 30)(11, 16, 26, 27)(15, 17, 23, 18)(24, 28)$
$(5, 8)(10, 25)(11, 23)(14, 22)(15, 26)(16, 17)(18, 27)(29, 31)$
$(3, 30)(4, 6)(5, 17)(7, 12)(8, 18)(10, 15)(11, 29)(14, 27)(16, 22)(20, 28)(23, 25)(26, 31)$
$(5, 23)(8, 11)(10, 16)(14, 26)(15, 22)(17, 25)(18, 29)(27, 31)$

The blocks of $\alpha(\mathcal{A})$ have two orbits under the action of this group, with orbit representatives:

$$\{0, 1, 2, 3, 6, 12, 19, 20\} \quad \text{(orbit of size 60)},$$
$$\{0, 1, 2, 5, 8, 14, 19, 22\} \quad \text{(orbit of size 560)}.$$

3. Polarity designs from $AG_{d+1}(2d + 1, q)$ for $q > 2$

We can modify Construction 2.1 for the case when $q > 2$. However, these modified designs do not typically have the same p-rank, nor the same intersection numbers, as the corresponding geometric design.

Let $\mathcal{A} = AG_{d+1}(2d + 1, q)$ for a prime power $q = p^s$. As before, let H be a hyperplane of $\mathcal{A}$ containing $\bar{0}$. For $q > 2$, $|H| < |X \setminus H|$, and so the outer and inner parts of any cross block will have different sizes. Thus, many of the special considerations in Construction 2.1 are unnecessary. The terminology from the binary case extends in natural ways. In particular, a block B is still either contained in H, or intersects H in q^d points. In the latter case, we still refer to B as a *cross block*.

The construction simplifies as follows:

Construction 3.1. Let α be a permutation of the affine d-spaces through $\bar{0}$ of the affine $2d$-space induced on H. Using α, we make the following alterations to the blocks of $\mathcal{A}$:

- If B is a block such that $B \subset H$ or $B \cap H = \emptyset$, we leave B unchanged.
- If $|B \cap H| = q^d$ and $\bar{0} \in B$, we replace the part of B equal to $B \cap H$ by $\alpha(B \cap H)$.
- If $|B \cap H| = q^d$ and $\bar{0} \notin B$, there is a block B_1 such that $\bar{0} \in B_1$, $|B_1 \cap H| = q^d$, and $B \cap H$ is a translate (or coset) of $B_1 \cap H$ in the group of translations of H, by considering H as a $2d$-dimensional vector space. Let $\{h_1 = \bar{0}, h_2, \dots, h_{q^d}\}$ be q^d distinct elements of H such that:
  - each coset of B_1 is represented exactly once in the set $\{B_1 + h_i \mid i = 1, \dots, q^d\}$, and
  - each coset of $\alpha(B_1 \cap H)$ is represented exactly once in the set $\{\alpha(B_1 \cap H) + h_i \mid i = 1, \dots, q^d\}$.

  By Lemma 2.2, such a set of h_i exists. We replace the part of B_1 equal to $B_1 \cap H$ with $\alpha(B_1 \cap H)$. For the coset of B_1 equal to $B_1 + h_i$, we replace the part equal to $(B_1 + h_i) \cap H$ with $\alpha(B_1 \cap H) + h_i$.

In particular, note that we no longer treat all blocks with the same inner part together. The outer parts of these blocks are not necessarily affine translates for $q > 2$.

Theorem 3.2. *The collection of blocks $\alpha(\mathcal{A})$ obtained from $\mathcal{A}$ via Construction* 3.1 *is a resolvable* 2*-design with the same parameters as $\mathcal{A} = AG_{d+1}(2d + 1, q)$.*

Proof. First note that, as in Construction 2.1, this construction preserves parallel classes, and so $\alpha(\mathcal{A})$ is resolvable.

We need to check that λ is unchanged. Let $P = \{x, y\}$ be a distinct pair of points in X.

- If $P \subset H$, then any block $B = B_{out} \cup B_{in}$ containing P has $P \subset B_{in}$. Because α permutes the inner parts of cross blocks, the number of cross blocks containing P is unchanged.
- Similarly, if $P \subset X \setminus H$, then the number of cross blocks containing P is unchanged.
- Suppose $x \in H$, $y \in X \setminus H$. Let B be a cross block containing x. Note that $\{B' \setminus H \mid B' \cap H = B \cap H\}$ partitions $X \setminus H$, and so exactly one such block contains $\{x, y\}$. Construction 3.1 preserves this property, and so the number of blocks with inner part $B \cap H$ containing $\{x, y\}$ is unchanged. Finally, for any block B, $\{x, y\} \subseteq B + h$ if and only if $\{x - h, y - h\} \subseteq B$, and so the counting does not change for cosets.

Thus we again have a design, although in this case we are only guaranteed a 2-design. □

Note that in this construction, we have specified that α permutes affine spaces. For $q > 2$, each point in our affine space is no longer identified with a unique point of a projective space, so we must make a small change in order to use a polarity of a projective space.

Let α be a polarity of the projective geometry $PG(2d - 1, q)$ induced on H. Then α permutes the projective $(d - 1)$-spaces in H. By viewing each point of $PG(2d - 1, q)$ as a 1-dimensional vector subspace, we can interpret each projective $(d - 1)$-space in H as an affine d-subspace containing $\bar{0}$. Thus α permutes the affine d-spaces of H containing $\bar{0}$, as required. Thus, it makes sense to speak of $\alpha(\mathcal{A})$. In this case, we can obtain more specific information about $\alpha(\mathcal{A})$.

Theorem 3.3. *If α is a polarity of the projective geometry $PG(2d - 1, q)$ induced on H, then the intersection numbers of the blocks of $\alpha(\mathcal{A})$ are congruent to* 0 *(modulo q).*

Proof. Any two blocks of $\mathcal{A}$ are either disjoint or share q^i points for some integer $1 \leqslant i \leqslant d$.

Let $B = B_{out} \cup B_{in}$ and $B' = B'_{out} \cup B'_{in}$ be cross blocks of $\mathcal{A}$, both containing $\bar{0}$. Construction 3.1 as applied to any block through $\bar{0}$ is equivalent to the construction of [8], and thus the intersection numbers of these blocks are unchanged. In particular, $|\alpha(B) \cap \alpha(B')| = |B \cap B'|$, and $|\alpha(B_{in}) \cap \alpha(B'_{in})| = |B_{in} \cap B'_{in}|$.

However, it is possible for the intersection numbers of cosets of cross blocks to change. In particular, it is not necessarily true (as it was for the case $q = 2$) that if two blocks share the same inner portion, then their outer portions are affine translates. They may be simply disjoint.

As before, $|\alpha(B_{in}) \cap \alpha(B'_{in}) + h| \in \{0, |B_{in} \cap B'_{in}|\}$, because the inner parts are affine subspaces. Note that $|B_{in} \cap B'_{in}| = q^j$ for some $0 \leqslant j \leqslant d$. If $|B \cap B' + h| = q^i$ for some $1 \leqslant i \leqslant d$, then $|B_{out} \cap B'_{out} + h| = q^i - |B_{in} \cap B'_{in}|$. Thus either $|B_{out} \cap B'_{out} + h| = q^i$, or else $|B_{out} \cap B'_{out} + h| = q^i - q^j = q^j(q^{i-j} - 1)$. It is clear that if $j \neq 0$, $|\alpha(B) \cap \alpha(B') + h|$ is a multiple of q. If $j = 0$, then as in the binary case, $|B_{in} \cap B'_{in} + h| = 1$ for all $h \in H$. Thus, $|B_{out} \cap B'_{out} + h| = q^k - 1$ for some $1 \leqslant k \leqslant d$, and so these blocks still intersect in a multiple of q points.

Finally, we consider the intersection of a cross block B and a non-cross block B'. Then $B \cap B'$ is entirely contained in either H or $X \setminus H$. If it is contained in H, then $B \cap B'$ is an affine subspace. By their dimensions, B and B' cannot intersect in only 1 point, so the size is a power of q. If the intersection is contained entirely in $X \setminus H$, then the intersection is unchanged by the construction. □

Example 3.4. The smallest example of a non-binary design is based on $\mathcal{A} = AG_3(5, 3)$, whose blocks may be viewed as the 3-dimensional vector subspaces of a 5-dimensional ternary vector space, and their cosets. The design $\mathcal{A}$ is a 2-(243, 27, 130) design with 10890 blocks. It is point- and block-transitive, with automorphism group $A\Gamma L(5, 3)$ of order $2^{10} \cdot 3^{15} \cdot 5 \cdot 11^2 \cdot 13$ (see for example [2]). Its 3-rank is 96, and the block intersection numbers are $\{0, 3, 9\}$.

The distorted design $\alpha(\mathcal{A})$, constructed with the orthogonal polarity of $PG(3, 3)$, has 82 point orbits, 1330 block orbits, and an automorphism group of order $2 \cdot 3^4$. There are 128 block orbits of size 1, 40 block orbits of size 6, and all remaining 1170 block orbits have size 9. Its 3-rank is 112, and the block intersection numbers are $\{0, 3, 6, 9\}$.

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